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COMMENT

On solutions of the Yang–Baxter equations without additivity

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Abstract. The relationship is described between solutions found by Ge and Xue and by the present author.

Recently two 4×4 solutions of the Yang–Baxter equations (YBE)

$$R_{12}(\lambda,\mu)R_{13}(\lambda,\nu)R_{23}(\mu,\nu)=R_{23}(\mu,\nu)R_{13}(\lambda,\nu)R_{12}(\lambda,\mu) \qquad (1)$$

were published [1]. The solutions are non-additive in the sense that $R(\lambda,\mu)\neq f(\lambda-\mu)$. The purpose of this comment is to generalize these solutions and describe their relationship to the solutions

$$R_{\rm V}(u,v)=\begin{pmatrix} u/v & 0 & 0 & 0\\ 0 & (uv)^{-1} & 0 & 0\\ 0 & 1-k & kuv & 0\\ 0 & 0 & 0 & v/u \end{pmatrix} \qquad k=\text{constant} \qquad (2)$$

$$R_{\rm VI}(u,v)=\begin{pmatrix} u_1/v_2 & 0 & 0 & 0\\ 0 & (u_1v_2)^{-1} & 0 & 0\\ 0 & W & -u_1v_2 & 0\\ 0 & 0 & 0 & v_2/u_1 \end{pmatrix} \qquad W=u_1/u_2+u_2/u_1 \qquad (3)$$

that are given in table 1 of [2] together with other non-additive solutions to the YBE. Note that the variables u, v in $R_{\rm VI}$ are two-component quantities.

To solve the equation (1) the ansatz

$$R(\lambda,\mu)=\begin{pmatrix} u_+(\lambda,\mu) & 0 & 0 & 0\\ 0 & p^{(+-)}(\lambda,\mu) & 0 & 0\\ 0 & W(\lambda,\mu) & p^{(-+)}(\lambda,\mu) & 0\\ 0 & 0 & 0 & u_-(\lambda,\mu) \end{pmatrix} \qquad (4)$$

was accepted (for easy orientation we use the notation of [1]). The ansatz can be justified either [1] by weight-conservation or [2] by the requirement that we look for solutions that for $\lambda=\mu$ are of the form

$$R=q\begin{pmatrix} 1 & 0 & 0 & 0\\ 0 & r & 0 & 0\\ 0 & 1-rt & t & 0\\ 0 & 0 & 0 & s \end{pmatrix} \qquad s=1 \text{ or } s=-rt. \qquad (5)$$

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An important fact exploited in the following is that the set of solutions of (1) is in general invariant under the transformations

$$R(\lambda, \mu) \mapsto \varphi(\lambda, \mu) R(\lambda, \mu) \tag{6}$$

$$R(\lambda, \mu) \mapsto [T(\lambda) \otimes T(\mu)] R(\lambda, \mu) [T(\lambda) \otimes T(\mu)]^{-1} \tag{7}$$

$$R(\lambda, \mu) \mapsto R(f(\lambda), f(\mu)) \tag{8}$$

where φ and f are scalar functions and T is a $GL(2)$-valued function.

We can exploit the symmetry (6) to set $u_+(\lambda, \mu) = 1$. Then we immediately get from (1) that $p^{(+-)}$, $p^{(-+)}$ are functions of one variable only

$$p^{(+-)}(\lambda, \nu) = p^{(+-)}(\lambda, \mu) = p^+(\lambda) \tag{9}$$

$$p^{(-+)}(\lambda, \nu) = p^{(-+)}(\mu, \nu) = p^-(\nu). \tag{10}$$

(cf (11), (12) in [1]). For $W(\lambda, \mu)$ we get the equation

$$W(\lambda, \mu) W(\mu, \nu) = W(\lambda, \nu)[1 - p^+(\mu) p^-(\mu)] \tag{11}$$

the general solution of which is

$$W(\lambda, \mu) = [1 - p^+(\lambda) p^-(\lambda)] \xi(\lambda) / \xi(\mu) \tag{12}$$

where ξ is an arbitrary function. The equations for $u_-(\lambda, \mu)$ then imply that

$$u_-(\lambda, \mu) = p^+(\lambda) q(\mu) \tag{13}$$

where

$$q(\mu) = 1/p^+(\mu) \qquad p^+(\mu) p^-(\mu) = k \in \mathbb{C} \setminus \{0\} \tag{14}$$

or

$$q(\mu) = -p^-(\mu). \tag{15}$$

The conclusion is that *there are just two solutions to the* YBE *(1) of the form (4).* They are

$$R_1(\lambda, \mu) = \varphi(\lambda, \mu) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & p^+(\lambda) & 0 & 0 \\ 0 & (1-k)\xi(\lambda)/\xi(\mu) & k/p^+(\mu) & 0 \\ 0 & 0 & 0 & p^+(\lambda)/p^+(\mu) \end{pmatrix} \tag{16}$$

$$R_2(\lambda, \mu) = \varphi(\lambda, \mu) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & p^+(\lambda) & 0 & 0 \\ 0 & W(\lambda, \mu) & p^-(\mu) & 0 \\ 0 & 0 & 0 & -p^+(\lambda) p^-(\mu) \end{pmatrix} \tag{17}$$

where W is given by (12) and p^+, p^-, φ and ξ are arbitrary functions.

The solutios in [1] are particular cases of (16), (17) where

$$p^+(\lambda) = q^{-1} \eta Q^{\sum_{j=1}^{m} \bar{\alpha}_j \lambda^j} \qquad p^-(\mu) = q^{-2}/p^+(\mu). \tag{18}$$

$$\varphi(\lambda, \mu) = q \varphi_+(\lambda, \mu) \qquad \xi(\lambda) = g(\lambda) \qquad k = q^{-2} \tag{19}$$

The appearance of functions φ and ξ in (16), (17) is a consequence of the symmetries (6), (7). *The symmetry (8) enables to consider* $p^{\pm}(\lambda)$ *and* $p^{\pm}(\mu)$ *as independent variables of the solutions.* This attitude was accepted in [2]. Namely, denoting $p^{+}(\lambda)=u^{-2}$, $p^{+}(\mu)=v^{-2}$ and choosing $\varphi(\lambda,\mu)=u/v$, $\xi(\lambda)=u^{-1}$, $\xi(\mu)=v^{-1}$ we get $R_1(\lambda,\mu)=R_{\mathrm{V}}(u,v)$. Similarly, denoting $p^{+}(\lambda)=u_1^{-2}$, $p^{-}(\lambda)=-u_2^2$, $p^{+}(\mu)=v_1^{-2}$, $p^{-}(\mu)=-v_2^2$ and choosing $\varphi(\lambda,\mu)=u_1/v_2$, $\xi(\lambda)=u_2^{-1}$, $\xi(\mu)=v_2^{-1}$ we get $R_2(\lambda,\mu)=R_{\mathrm{VI}}(u,v)$.

References

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[2] Hlavatý L 1987 *J. Phys. A: Math. Gen.* **20** 1661